Explicit stationary state solution of the Oslo model

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under supervision of Vincent Rossetto

Plan of the talk

- Motivations
- The Oslo model
- State of the art
- Main results
- Perspectives

Based on: Valentin Lallemant et Vincent Rossetto. *An explicit formula of the Oslo stationary state*, Août 2025. arXiv:2508.06315 [cond-mat] (submitted to JSTAT).

Historical motivations

<u>Problem:</u> Power-law behaviors in non-equilibrium systems are common.

<u>Attempt for a solution:</u> Self-Organized Criticality (SOC)

Stationary non-equilibrium systems;

Intermittent dynamics;

Display observables with scale invariant features.

Genesis of SOC: *Per Bak, Chao Tang, and Kurt Wiesenfeld. "Self-Organized Criticality: An Explanation of 1/f Noise". In: Physical Review Letters 59.4 (June 1987).*

Oslo model origin

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Tracer Dispersion in a Self-Organized Critical System

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(Received 25 January 1996)

We have studied experimentally transport properties in a slowly driven granular system which recently was shown to display self-organized criticality [Frette *et al.*, Nature (London) **379**, 49 (1996)]. Tracer particles were added to a pile and their transit times measured. The distribution of transit times is a constant with a crossover to a decaying power law. The average transport velocity decreases with system size. This is due to an increase in the active zone depth with system size. The relaxation processes generate coherently moving regions of grains mixed with convection. This picture is supported by considering transport in a 1D cellular automaton modeling the experiment.

Real experiment consistent with SOC definition

Oslo model

Theoretical perspectives

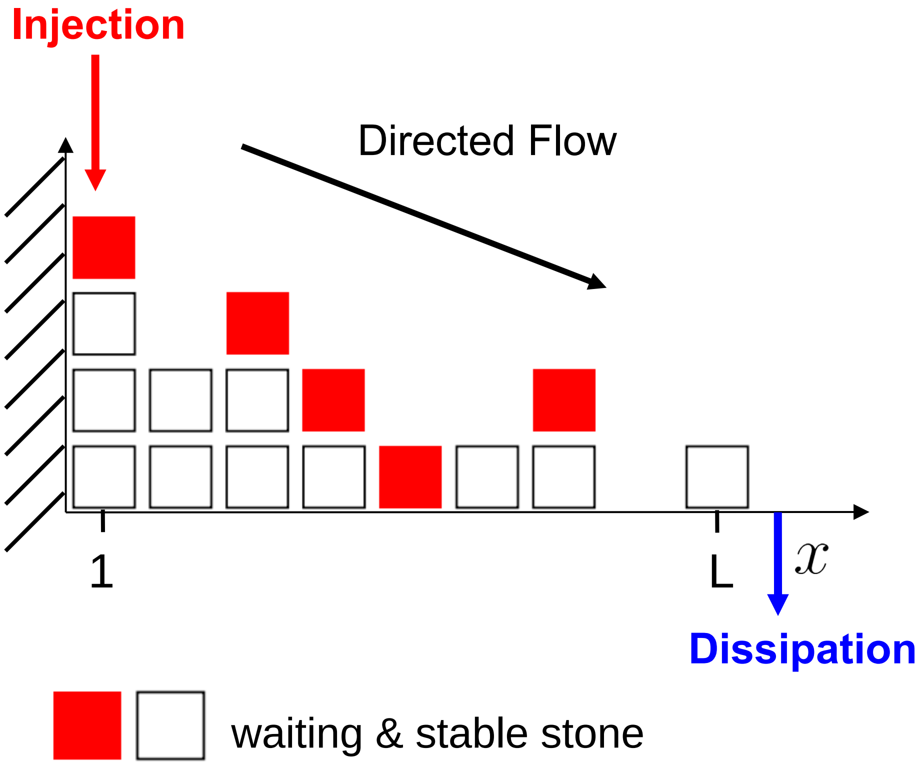
- Stationary state is **a prior to any computation of the stationary properties** in principle (avalanche spectrum moments, n-points correlations, …);
- **Completely characterized** in a related model: the **Totally Asymmetric Oslo Model (TAOM)**, natural directed variant of this model;
- Yet, it is **not known** for the Oslo model;
- **Exact results** on SOC systems are **valuable** given the few that exist.

TAOM:

Gunnar Pruessner. "Exact solution of the totally asymmetric Oslo model". In: JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL (June 2004)

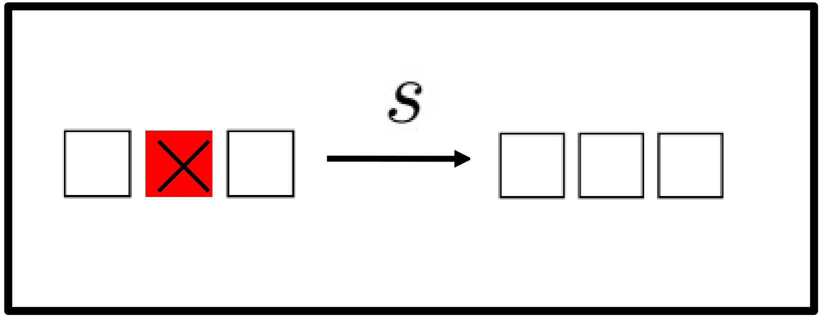
Thordur Jonsson and John F. Wheater. "Area Distribution for Directed Random Walks" In: Journal of Statistical Physics (May 1998)

The Oslo model in its g-representation



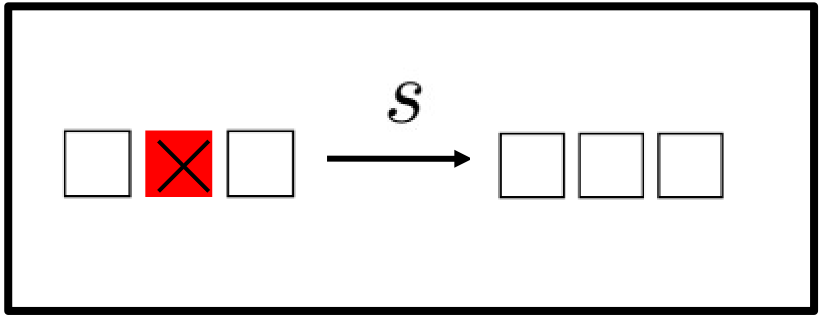
Instructions

Deterministic settling

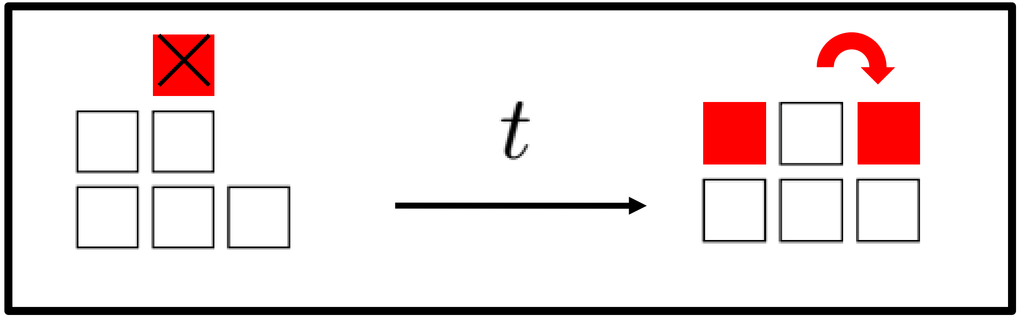


Instructions

Deterministic settling

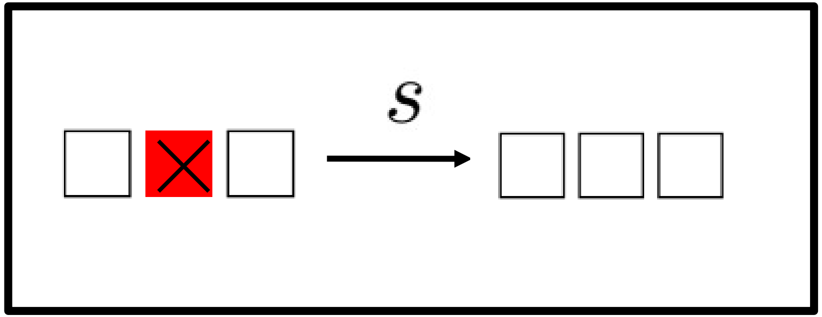


Deterministic toppling



Instructions

Deterministic settling



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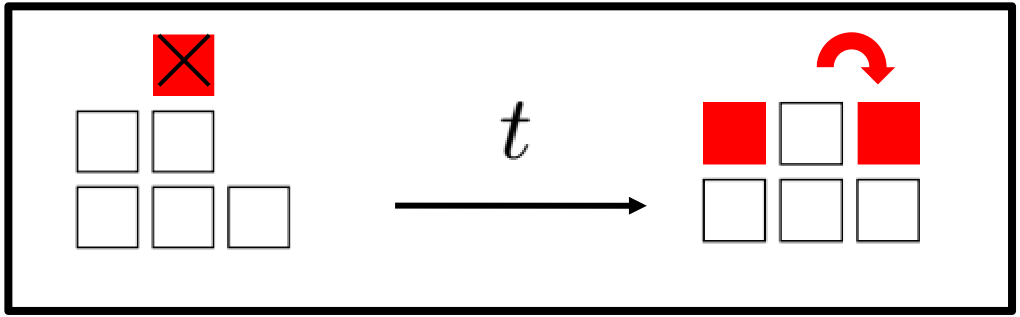
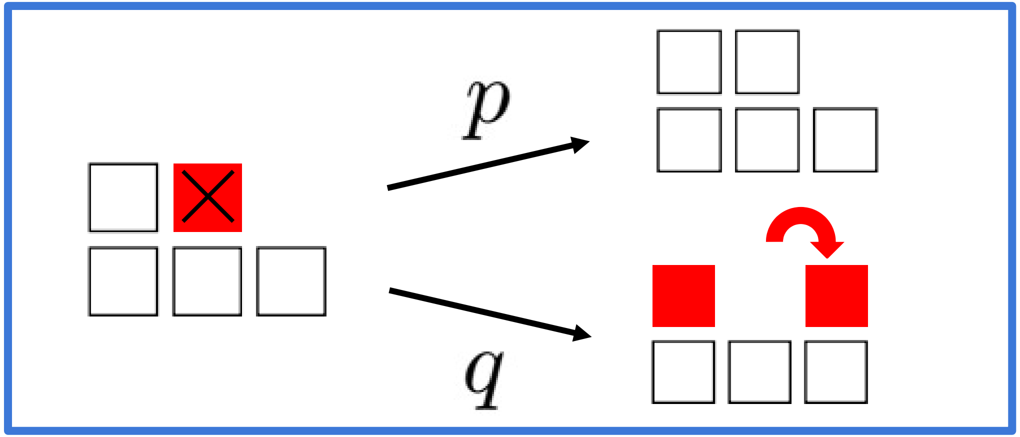


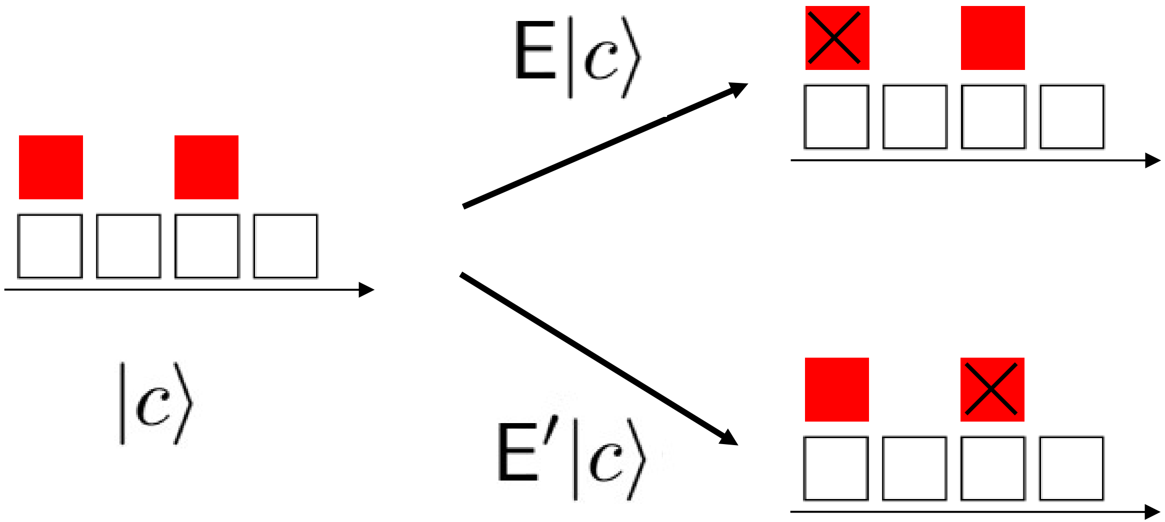


Stochastic settling

Stochastic toppling

Which stone to select?

Set of operators $\mathcal{E}$ such that for two different $\mathrm{E}, \mathrm{E}' \in \mathcal{E}$



+ perform corresponding instruction(s)

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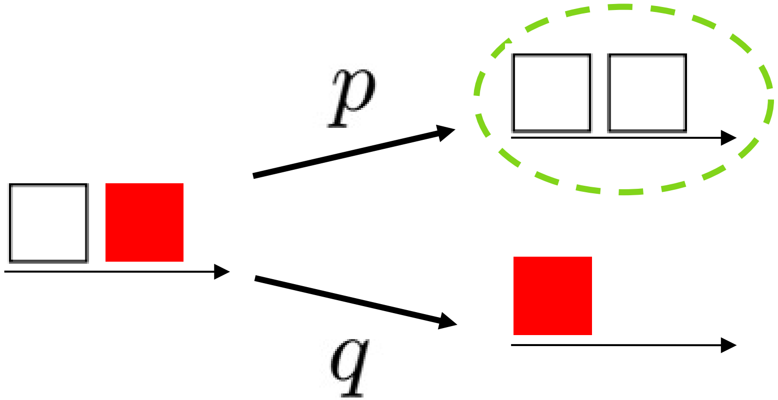
Stabilization

Iterate action of a given $\mathsf{E} \in \mathcal{E}$ until a stable state (linear combination of stable configurations) is reached



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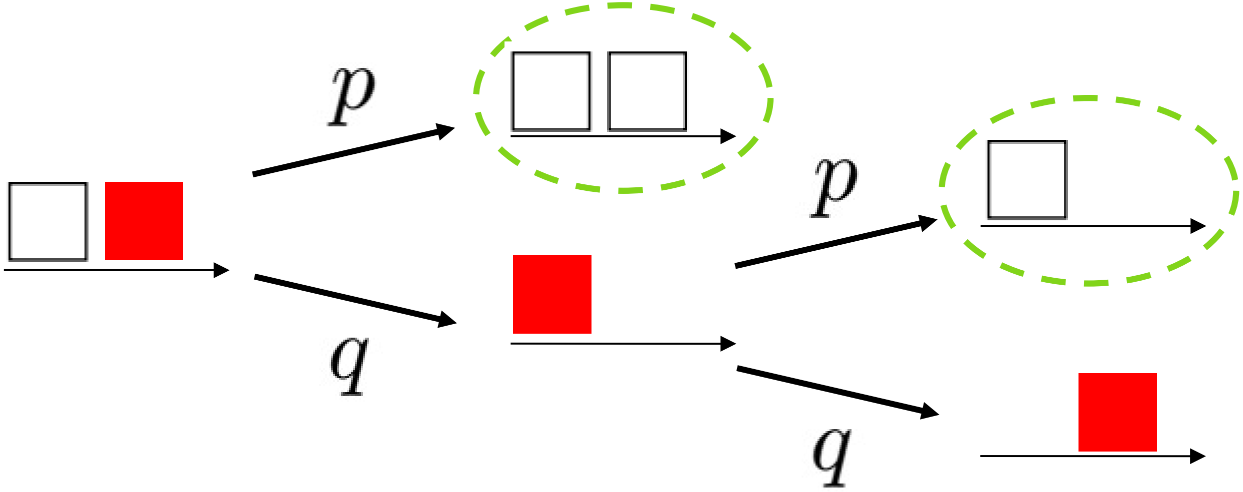

Stabilization

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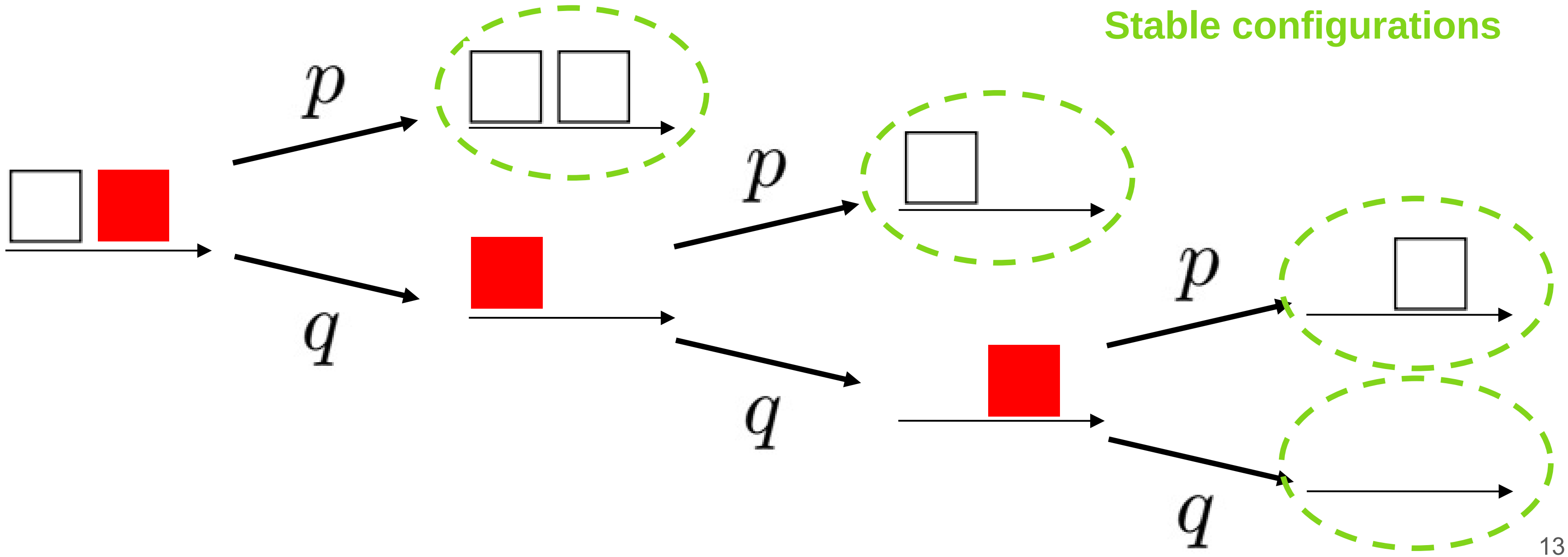


Stationary state

Driven dissipative setup:

Inject one stone ⟶ Stabilize ⟶ Restart.

Defines a **Markov operator** W and **stationary state** $|\psi\rangle$ such that

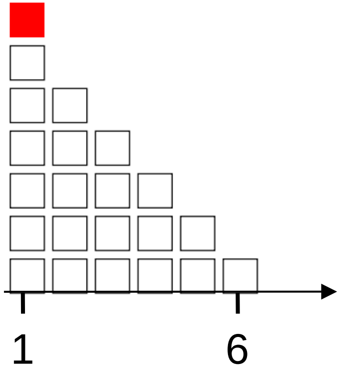
$$\mathsf{W}|\psi\rangle = |\psi\rangle = \sum_{r\in\mathcal{R}} P_r |r\rangle$$

Where $\mathcal{R}$ is the **set of recurrent configurations** where

$$\mathrm{card}(\mathcal{R}) \sim \left(\frac{1+\sqrt{5}}{2}\right)^{2L} \sim 2.6^L$$

State of the art

- The model is **Abelian with respect to stabilization. Any** evolution operator leads to the **same stationary state!**
- **Stabilization** always ends after a **finite number of time steps**.
- The stationary state can be obtained **injecting one stone on the maximal stable configuration and stabilizing.**



Deepak Dhar. *Steady State and Relaxation Spectrum of the Oslo Rice-pile Model. In: Physica A: Statistical Mechanics and its Applications (Sept. 2004).*

New results

- → Their exists an **equivalence relation** on the set of recurrent configurations that allows to **factorize the stationary state**.

- → To each recurrent configuration is associated a **set of invariants.**

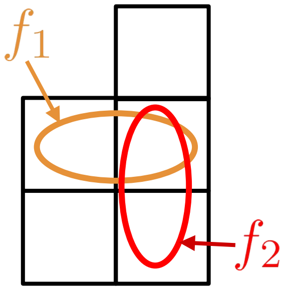
- → The **stationary probability** of a recurrent configuration calculation is mapped on a **coloring problem** specified by the invariants.

We have now an explicit formula of the stationary state!

The coloring problem

To each recurrent configuration, we can associate **a discrete domain** $\mathcal{T}_r$ (set of sites) and a **set of subdomains** $\mathcal{F}_r$ also called **constraints**.

Example:



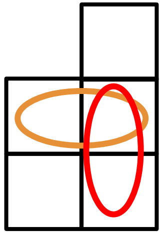
Problem with $\text{card}(\mathcal{T}_r) = 5$ and two constraints f_1 and f_2

Computing the stationary probability corresponds to solve the following combinatorial problem:

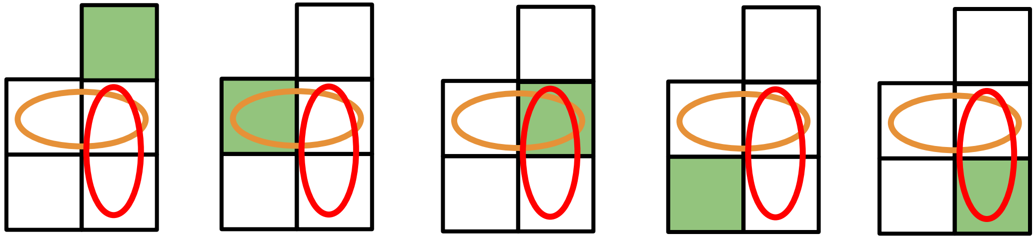
In how many ways can we color *i* sites in green without filling completely a constraint?

Direct counting

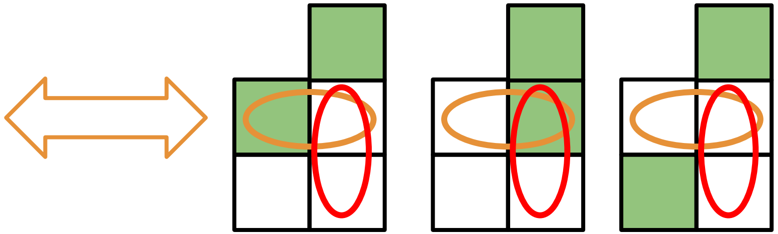
$$\gamma_0 = \binom{N}{0} = 1 \Longleftrightarrow$$

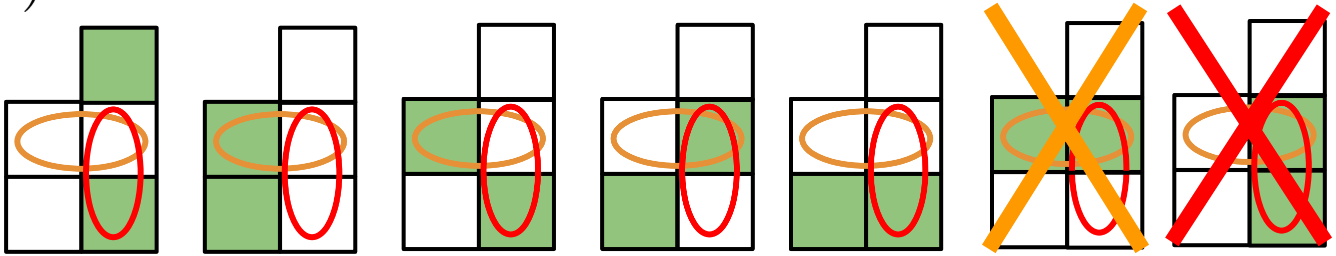


$$\gamma_1 = \binom{N}{1} = 5 \Longleftrightarrow$$



$$
\begin{aligned}
\gamma_2 &= \binom{N}{2} - \left(\overbrace{\binom{2}{2}\binom{N-2}{0}}^{f_1} + \overbrace{\binom{2}{2}\binom{N-2}{0}}^{f_2} \right) \\
&= 10 - (1 + 1) = 8
\end{aligned}
$$



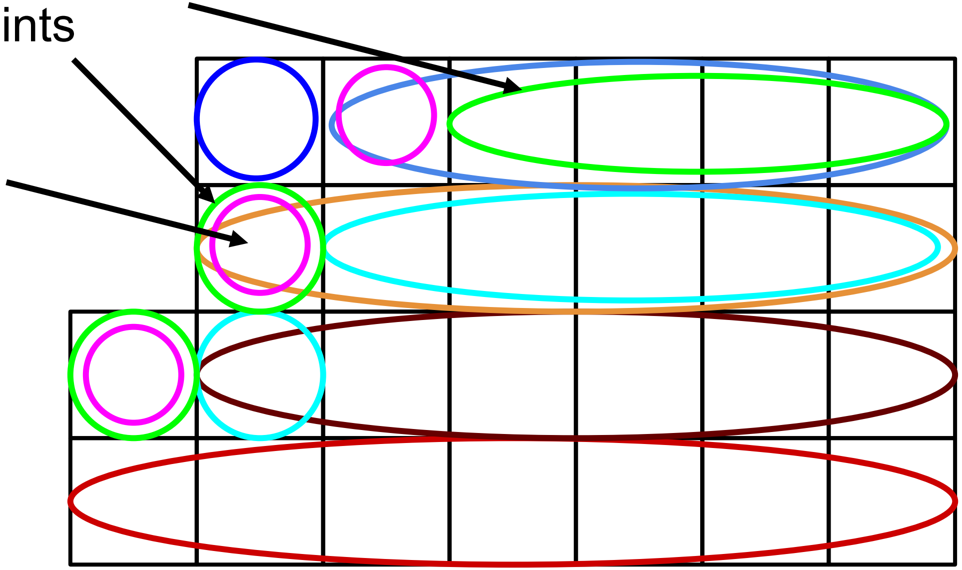


General solution: inclusion-exclusion principle

Simple realistic case in the Oslo model

Non-connected constraints

Multiple intersections of the constraints



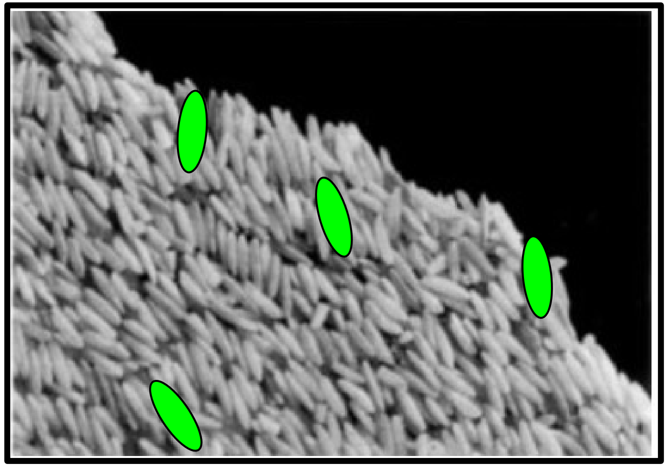
Perspectives on this work

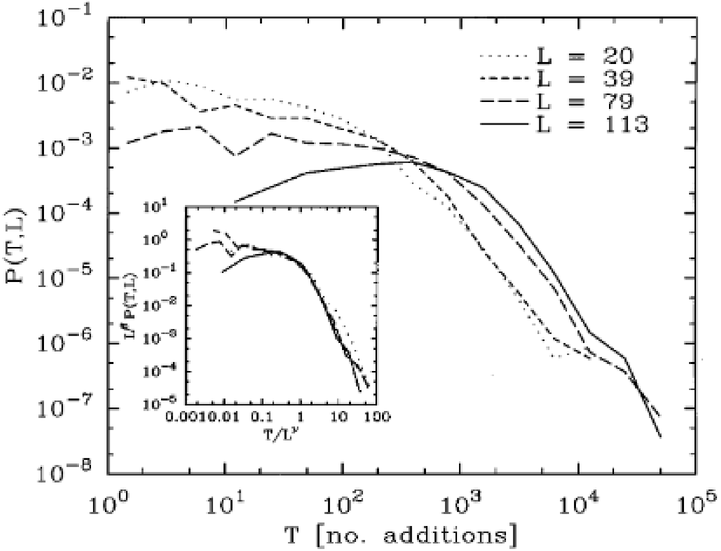
- Understand the **spatial structure** of the constraints in the coloring problem.
- Evaluate the **scaling limit** of the stationary state, **develop approximation schemes**.
- **Extend** the approach to **compute stationary observables**: global and local average densities, avalanche moments, etc.

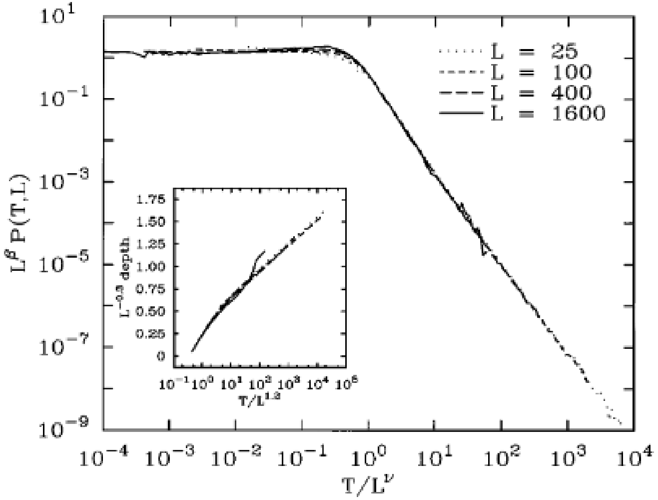
Thank you for your attention.

Annexes

Picture of the experimental setup.

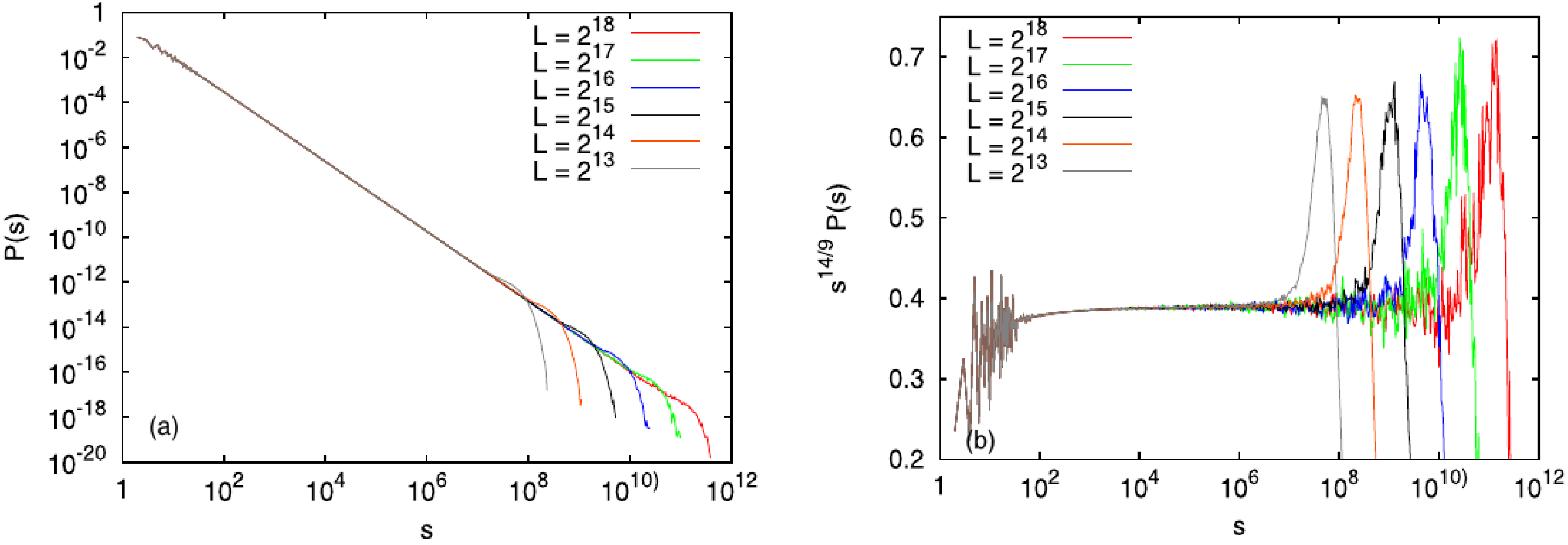






Experimental results for the transit time (up) and numerical results for the Oslo model (left).

Extended studies on avalanche spectrum



Plots from: *Peter Grassberger, Deepak Dhar, and P. K. Mohanty. “Oslo model, hyperuniformity, and the quenched Edwards-Wilkinson model”. In: Physical Review E (Oct. 2016).*

Equivalence relation on $\mathcal{R}$

The stationary state **factorizes**

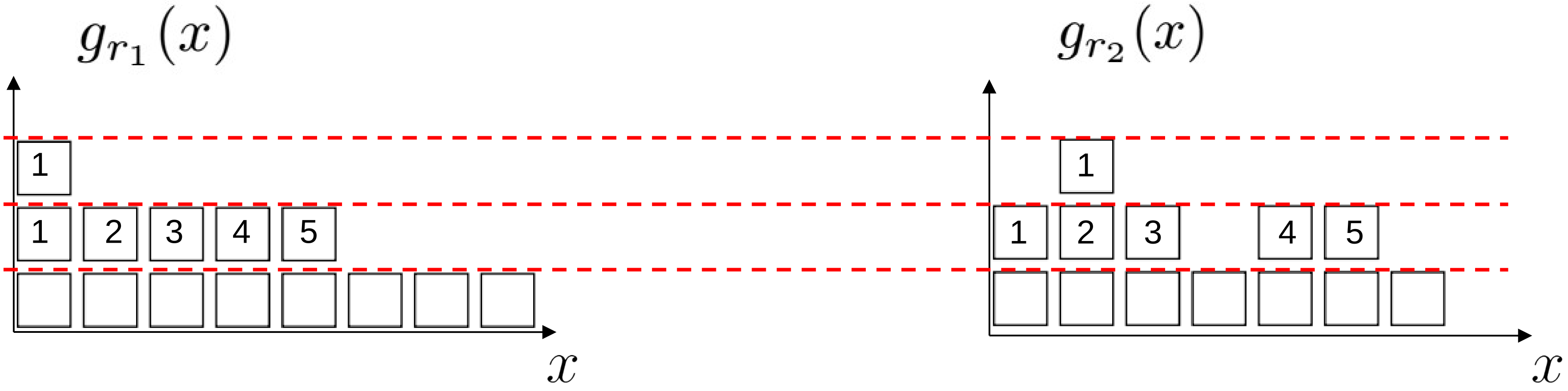
$$|\psi\rangle = \sum_{\tilde{r} \in \mathcal{R}_{\text{nat}}} P_{\tilde{r}\uparrow}(p,q) \left(\sum_{r \in [\tilde{r}]} p^{\pi_r} q^{\theta_r} |r\rangle \right)$$

Complexity reduction of the problem

$$\text{card}(\mathcal{R}) \sim 2.6^L >> \text{card}(\mathcal{R}_{\text{nat}}) \sim 2^L$$

Geometry behind the equivalence

Equivalent configurations share the **same number of stones at each constant height level**, except potentially for the lowest one.



$$r_1 \sim r_2$$

Stationary state for L=3

|222> (p^3)
+|122> (3.p^4q+ 3.p^3q^2+ p^2q^3)
+|212> (6.p^5q^2+ 9.p^4q^3+ 5.p^3q^4+ p^2q^5)
+|022> (6.p^5q^3+ 9.p^4q^4+ 5.p^3q^5+ p^2q^6)
+|221> (6.p^5q^3+ 9.p^4q^4+ 5.p^3q^5+ p^2q^6)
+|112> (18.p^5q^4+ 33.p^4q^5+ 24.p^3q^6+ 8.p^2q^7+ pq^8)
+|012> (18.p^5q^7+ 33.p^4q^8+ 24.p^3q^9+ 8.p^2q^10+ pq^11)
+|021> (18.p^5q^8+ 33.p^4q^9+ 24.p^3q^10+ 8.p^2q^11+ pq^12)
+|102> (18.p^5q^9+ 33.p^4q^10+ 24.p^3q^11+ 8.p^2q^12+ pq^13)
+|111> (18.p^4q^10+ 33.p^3q^11+ 24.p^2q^12+ 8.pq^13+ q^14)
+|121> (18.p^5q^5+ 33.p^4q^6+ 24.p^3q^7+ 8.p^2q^8+ pq^9)
+|202> (18.p^6q^6+ 33.p^5q^7+ 24.p^4q^8+ 8.p^3q^9+ p^2q^10)
+|211> (18.p^5q^7+ 33.p^4q^8+ 24.p^3q^9+ 8.p^2q^10+ pq^11)